The Roles of Mesopotamian Bronze Age Mathematics
Tool for State Formation and Administration – Carrier of Teachers' Professional Intellectual Autonomy

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Tilegnet Naïm
som velkomst

1. Uruk: A mathematical state

Around the mid-fourth millennium,[1] changing ecological conditions made possible the introduction of large-scale artificial irrigation in the south of modern Iraq. A direct consequence of this was a demographic explosion, an indirect consequence the emergence of a social structure characterized by several levels of administrative control and extensive division of labour – a genuine *state*, anthropologists would say, centred around the great temples. To all we know it was the earliest state in human history.

Excess of food and population did not in themselves create this state; even the obvious need for the organization of exchange with other regions could not do that (the zone was rich in grain, dates, fowl, fish – and bitumen and mud; all other resources had to be imported).[2] However, tools were at hand which allowed the transformation of needs and economic-demographic occasion into driving forces.

Most important of these tools was an age-old accounting system based on small tokens of burnt clay, shaped as small and larger spheres, cones, discs, cylinders, etc. From the eighth millennium onward, these tokens were in use in Anatolia, Syria, Iran and Iraq. According to what can be inferred from the archaeological context where the tokens are found and from their later influence on the writing system they represented standard quantities of grain, oil and other goods, and heads of livestock; they served within accounting-based, religiously legitimated food redistribution systems at village-level. Being responsible for the management of redistribution and accounting was apparently a role that carried the highest social prestige. (Evidently, the system need not have functioned in the same way in all places where it was in use.)

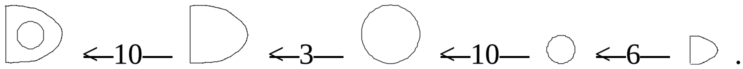
In the earlier fourth millennium, as a social system of regional extension took shape in a valley area east of Mesopotamia, the accounting system acquired new dimensions. Tokens were now put into closed spherical clay containers ('bullae'), which served as 'bills of lading' for goods delivered from the periphery to the temples of the central city Susa. On the surface of the bulla the cylinder seal identifying the responsible official or instance was rolled; since the bulla could not be 'read' without losing its documentary value, surface-impressions corresponding to the tokens that were contained might also be made. As a response to a need for finer distinctions, the number of token-types was increased, for instance (but not exclusively) through strokes at their surface.

During the archaeological 'phase V', this system was taken over in Uruk, the central city of the emerging Mesopotamian state, and it was realized that with impression of the tokens on the surface of the bullae the tokens themselves were superfluous, and that a flat clay surface carrying only the impressions and the seal (a so-called 'numerical tablet', actually rather metrological) would serve just as well. During the ensuing 'Uruk IV' phase, the accounting system was expanded into or integrated in a new invention: writing. Whether Uruk IV writing should be seen as a violent expansion of the old system or a new invention into which the

[1] BCE, as all dates in the following. I follow the so-called 'middle chronology'.

[2] Contrary to what is sometimes maintained (and claimed to be the motive force behind the emergence of mathematics), irrigation itself was in no need of central planning.

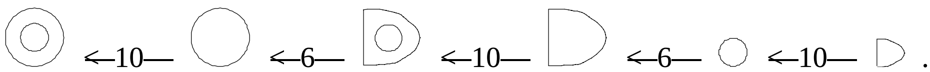
old system was integrated is a heated but rather superfluous discussion; grossly speaking, in any case, the script had two components; pictographic word signs traced with a pointed stylus on the surface of a clay tablet, and metrological signs impressed either vertically or obliquely and by either the thin or the thick end of a cylindrical stylus, rendering thus the profile of various tokens (primarily spheres and cones).[3] The most important metrological sequence was the grain system, which can be rendered thus (in a notation due to Jöran Friberg):



A small 'sphere' is thus equivalent to 6 small 'cones', etc. The whole sequence builds on fixed numerical ratios and bundling. In contrast, the symbols used in the 'numerical' tablets from Uruk V and the tokens to which they correspond appear to have corresponded to actual standard containers with no exactly fixed numerical ratios; at least, "bundling" was no principle.

The introduction of numerical exactness is mentally related to another apparent innovation: a notation for abstract numbers. Only when the support of a tablet could guarantee lasting physical proximity of distinct signs for quantity and quality was it possible to separate the two, for instance by writing the sign for "2" alongside a crossed-over circle meaning "sheep" (probably a drawing of the corresponding token, a marked flat circular disk); as long as sheep were represented by freely rattling disks, each of these had to represent "1 sheep".

The number sequence looks like this:



Here, the small cone stands for 1, the small sphere thus for 10, and the large cone for 60. The sign for 600 may have been meant multiplicatively, as a large cone (60) multiplied by a small sphere (10). The cylindrical stylus having but two ends, an even larger cone could not be used for 3600, but the multiplicative suggestion turns up again in the construction of 36000 (and in that of the largest grain unit, see above).

Two further observations could be made on the number sequence. Firstly, though no place-value system it is already sexagesimal, that is, with base 60 – more precisely, it is alternatingly sexal-decimal. Secondly, although its signs are shared with those of the grain sequence, not only the factors but even their order is different. This has two implications. Firstly, it was no free creation from scratch but probably bound by a pre-existing oral system (which however need not have gone as far as 3600); if not, why change the factors? Secondly, the choice of the large cone instead of the large sphere for 60 shows that the inventors were already conscious that 60 was a 'larger unity', that is, of the sexagesimal principle.[4]

Also of interest is the area metrology. Names for the units (and other evidence) suggest

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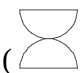
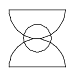
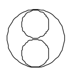
[3] In the third millennium, first the word signs, later also the metrological symbols were stylized and made by oblique (whence wedge-shaped) impressions of a prismatic stylus; in this way the script became 'cuneiform'.

[4] A particular 'bisexagesimal' number sequence with units 1, 10, 60, 120 (), 1200 () e 7200 () was used for particular purposes. It was probably created in order to facilitate specific administrative procedures; it is thus evidence of flexible thinking, not of thinking "not yet ready for abstraction".

that these had originated as ‘natural' (irrigation and seed) measures. In Uruk IV, however, they were geared to the length metrology, which permitted that areas could be calculated in terms of the linear dimensions of fields.

All of this already presupposes systematic mathematical thought. Similar thought is seen in the creation of a general category ‘sub-unit': sub-units are formed in the same way in all metrologies (by turning the smallest unit 90° clockwise – once again a possibility only offered by the fixation in clay).

That mathematical thinking entered the creation of the script should not astonish, since the whole purpose of this creation was to apply mathematics. Some 85% of all texts from Uruk IV and the ensuing Uruk III period are indeed accounting texts (including some pseudo-accounting or ‘model documents' meant for schooling purposes, characterized by too round or too large numbers and the absence of the seal of a responsible official). The rest are sign lists serving the training of the script (with some 1000 signs in use, training was certainly needed!).

However, mathematical thought was important not only in its shaping of the bureaucratic tools of the temple bureaucracy; it shaped the statal structure itself.

For this we have no direct testimonials. However, much can be concluded from indirect arguments. At the most general level, later literary evidence (royal boasting etc.) shows that the transformed continuation of the old token system went together with an equally transmuted continuation of the principle inherent in economic redistribution: social justice. Ideologically, the traditional bond between redistribution and accounting ensuring its justness reemerged, or so it seems, as an ideology of social justice understood as *(mathematically) just measure*.[5] In agreement with this principle, land was allotted to high officials in precise proportions corresponding to their rank, and foodstuffs and other necessities provided to workers and cattle in precisely measured rations. At both levels, the new mathematical technologies (the area metrology and the written accounting) were necessary preconditions.

But even the way the social structure was understood was influenced by ‘bureaucratic mathematics'. Accounts were organized in tabular form, roughly speaking in cartesian product. One of the sign lists, the one recording the various professions present in the state, is made in precisely the same way (not graphically but in contents); along one dimension, it catalogues the various professional fields; along the other, the three ranks – head, overseer, common worker. To us this may seem the obvious way to make such a list; but our ways to perceive structures is also shaped by seeing much information in tabular form. (Actually, our tabular form is, through Late Babylonian, Hellenistic and medieval astronomical tables, inherited directly from the Mesopotamian administrators.)

The close bond between mathematics and bureaucratic state holds both ways. To all we know the literate and numerate class coincided with the class of temple managers; no autonomous profession of scribes seems to have existed. Concomitantly, mathematical thought

[5] We should not be taken in by this pretence of justice. Mathematical justice could be just as cruel as the ‘surgical mathematical precision' of present-day bombing. The favourite theme of the cylinder seals of the Uruk officials is a high priest observing overseers who beat up pinioned prisoners, probably teaching them to be obedient slave workers. In many respects, the Uruk state was probably similar to the plantation economies of the Early Modern West Indies (also well supplied with managers and accountants taking care that slaves were exploited ‘rationally').

had no autonomy from its bureaucratic functions. Just as absent from the written record as non-bureaucratic uses of the script (either for ‘literary' or ‘propaganda' purposes) are mathematical play and investigation of the formal properties of numbers or geometrical shapes. Uruk statehood, Uruk writing and Uruk mathematics appear to have formed a single inextricable complex.

2. The Sumerian third millennium

During the first half of the third millennium, a polycentric system of city-states ruled by military leaders (‘Kings') evolved in southern Iraq. After a couple of centuries that have left us no written documents the script turns up again, now provided with sparse grammatical elements that allow us to identify the language as Sumerian; names present in some of the documents show, however, that Akkadian[6] speakers were already present. After 2400 followed a phase of centralization, and around 2350 Sargon, an Akkadian chief, subdued the whole of central and southern Iraq. His successors at first extended the empire as far as Susiana, Syria and Anatolia, but around 2200 it collapsed. After a century a new phase of centralization followed under the ‘Third Dynasty of Ur' (2112 to 2004), which subjected the south and the centre to a direct administration to which we shall return, and the north and Susiana to less direct and less lasting rule.

All these political developments influenced mathematics. At the most general level, two trends are conspicuous: firstly, extensive ‘sexagesimalization', that is, creation of higher and lower metrological and numerical units through multiplication and division by 60; secondly, adaption of metrologies to administrative procedures (overruling when needed the principle of sexagesimalization). Both trends continue tendencies that were present in the fourth millennium.

Yet fundamental innovation can also be observed, socially as well as epistemologically. In Shuruppak (c. 2600) and contemporary cities, written contracts and administrative documents exhibit a complex social structure, within which the scribes had a pivotal role and formed a true profession. Many school texts have also been found. Some descend from the Uruk lists, others constitute quite new types: literary texts (a proverb collection, a hymn) and mathematical texts not directly linked to practice even if dressed as if they were – we may speak of them as ‘supra-utilitarian'. One example: A silo containing 40×60 gur (‘tuns') of grain, each of 8×60 sila (‘litres') is distributed in portions of 7 sila per worker. The answer (164571 workers, and a remainder of 3 sila) is found correctly in one tablet, while an error in a parallel tablet reveals the method (division of 2400 gur by 7, ensuing multiplication of the result by 480, conversion of the remainder of 6 gur into sila, etc.).

The problem belongs to an apparently favourite genre: the division of very large round numbers or quantities by difficult divisors. Other tablets suggest an emerging interest in ‘interesting' geometrical configurations – e.g., four circles inscribed in a square.

The use of the script for literary purposes and the solution of supra-utilitarian problems may be understood as ways to test the two main professional tools and to affirm professional identity – a *true scribe* was one whose skill to read, write and calculate exceeded that which was needed in administrative practice; it could only be appreciated by other scribes.

In the following centuries, political powers discovered the utility of literature for

[6] Akkadian is a Semitic language, whose later main dialects are the Babylonian and the Assyrian.

propaganda purposes; Sargon in particular had the religious hymns rewritten so as to be adequate for his political project. Supra-utilitarian mathematics had no such potentialities. What we find in the Sargonic epoch is thus, firstly, an effort to homogenize metrologies so as to fit the extended accounting procedures of the new realm: e.g., introducing a ‘royal gur' of 300 sila, a value that facilitated certain administrative procedures. Further, continued teaching of supra-utilitarian mathematics in school – for example (an example that will turn out to be important) to find one dimension of a rectangle if the area and the other dimension are known. Given the metrologies, this is no mere division problem;[7] nor is it a problem which a surveyor would encounter in any daily practice. One tablet contains an example of more sophisticated geometrical knowledge of no practical use: that a trapezium is bisected by a parallel transversal whose square is the average between the squares on the parallel sides.

It is probable but not certain that these geometric riddles were adopted into the school (where everything was done in Sumerian) from an Akkadian environment of ‘lay' (that is, non-scribal) surveyors. What *is* fairly certain is that they were transmitted in subsequent centuries by an environment of this kind. Since mensuration depended on metrologies once created by the administrators of the fourth millennium it is obvious that the ultimate inspiration of the ‘lay' profession was the ‘learned' world. We do not know with certainty when the two separated; land sale contracts from Shuruppak mention a surveyor alongside the scribe. Perhaps the surveyor was a specialized scribe, perhaps he belonged to a lay group; in this latter case the lay surveyors’ profession would be coeval with the autonomous scribal profession.

Presumably, the types of mathematics we know from the Sargonic era did not disappear with the empire, even though we have little evidence from the city-states of the following century. With Ur III, on the other hand, fundamental change set in.

The decisive moment was not the conquest of regional power in 2012 but a military reform in 2074 immediately followed by an administrative reform (certainly already prepared). In what was perhaps introduced as a state of emergency but which became permanent, the large majority of workers from the centre of the empire were organized in troops under overseer-scribes, who were responsible for the work of their unit calculated in accordance with fixed norms (so much dirt dug out, so much cloth woven in one day, etc.) and converted into abstract accounting units (at times $^{1}/_{60}$ of a working day, at times weight units of silver or volumes of grain). For each month (also if the real length was 29 days), the overseer had to press 30 days of work from his labourers. Labourers lent to colleagues and those who were ill or dead or had run away were entered into the account as credit, those borrowed as debit. The deficit of the scribe was accumulated from year to year (surplus is extremely rare), and at his death the family was made responsible; if they could not pay, widow and children might be put into the working troops.[8] Museums possess tens of thousands of the accounting tablets that resulted.

The detailed calculation of a national economy is, and was, no easy matter. Additions and subtractions could be made by traditional means – at least since the mid-third millennium, some kind of abacus (called “the hand” in agreement with its possession of five sexagesimal levels) had been in use. Difficult were the numerous multiplications and divisions, not least

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[7] A parallel would be to find the length of a rectangle with area 2500 acres if the width is $7^{1}/_{5}$ inch.

[8] Robert Englund, the Assyriologist who deciphered the system, spoke of it as a ‘Kapo economy'.

because many of these would involve metrological sequences that were not completely sexagesimal – the nindan (‘rod', of c. 6 m), the basic unit for (horizontal) distance, was thus subdivided into 12 kush (‘cubits'), each of these being 30 ‘fingers'.

Simplification of the task asked for several innovations. First of all, for intermediate calculations a ‘floating-point' sexagesimal place-value system was created – that is, the number that is transliterated 7.30 might stand for 7×60+30 = 450 (then we use the transcription 7,30), but also $7+{}^{30}/_{60} = 7^{1}/_{2}$ (transcription 7;30), ${}^{7}/_{60}+{}^{30}/_{3600} = {}^{1}/_{8}$ (transcription 0;7,30), etc. Next tables of reciprocals were produced (by means of these a division by *n* could be reduced to a multiplication by ${}^{1}/_{n}$) and of the products of important numbers (not least those appearing in the table of reciprocals) with the numbers 1, 2, 3, ..., 19, 20, 30, 40, 50. ‘Metrological tables' expressed the various metrological units in terms of a basic unit – for instance, a kush as 5 (i.e., 0;5 nindan), and a ‘finger' as 10 (i.e., 0;0,10 nindan); finally, tables of ‘fixed factors' listed technical constants, e.g., how much dirt a worker should excavate in a day.

The important step was not to *invent* the place value system; probably there was no need for an invention, since the principle was inherent in the abacus. However, without the whole array of tables being at the disposition of every scribe (at best memorized), the place value system would have been of no use; its implementation must have asked for a decision at the highest level and for a curriculum reform in the scribe school.

This reform, it turns out, also involved an elimination. The only mathematical Ur-III school texts that have been found are model documents, as in fourth-millennium Uruk. Even these are few, and the absence of supra-utilitarian problems from the archaeological record might be due to the accident that no Ur-III school room has been excavated. It is more telling that the whole vocabulary needed to formulate mathematical *problems* was reconstructed in the following period, as if it had not existed in Sumerian (even though it *had* existed in the Sargonic school). It appears that even that modicum of intellectual liberty that is needed for the solution of mathematical problems was not granted future overseer-scribes.

At the same time, *the scribe* was exalted. Shulgi, the king who undertook the reforms, boasts in hymns written in his honour (he proclaimed himself divine) that he mastered the whole scribal art with eminence. However, his mathematics does not go beyond “addition, subtraction, counting and accounting”; since he boasts in other domains of all imaginable feats it appears that his ghost-writers knew of no other mathematics.

Shulgi also boasts of being justice personified. Apart from century-old commonplaces, his justice turns out to be that of accounting and of the metrological reform. As the Uruk-IV state appears to have done, Shulgi’s state was legitimized as a manifestation of justice, its justice consisting in exploitation (extreme exploitation, as the careful registration of food rations and workers’ mortality demonstrates) being not arbitrary but mathematically controlled; once more, the recent commonplace of ‘surgical precision' comes to mind.

3. Old Babylonian culmination

The Ur-III empire did not last; one reason may have been the costs of its bureaucracy. Already in 2025 the peripheral regions made themselves free, and some twenty years later the centre fell into small city states. Gradually, some of these absorbed the others, and in the eighteenth century Hammurapi of Babylon subdued the complete south and centre; from this moment onward we may speak of the whole region as ‘Babylonia'.

Nor did the anticipation of ‘scientific management' last. The Old Babylonian period is characterized by individualism, both at the economic level (even though it is too early to speak of a general market economy) and at the level of culture. Land, instead of being worked by labour troops, was often farmed out on contract. Private commercial and personal correspondence emerges, giving rise (together with private accounting) to the private employment of scribes and to the appearance of free-lance scribes; the seal, until recently an indication of the office, became an identifier of the person. We may speak of the advent of an *ideology of personal identity*.

For our purpose, what is important is the manifestation of this ideology in the scribal environment – a particular manifestation which we know rather well from texts used in school to impart the idea of what characterizes a *true* scribe.

The Sumerian language was dead, and Babylonian could be written with a phonetic syllabary of no more than 70 signs. However, a true scribe also used numerous word signs – coming from Sumerian, but now pronounced in Babylonian (as the ‘word sign' *viz* for Latin *videlicet* is pronounced *namely* in English). But this was not enough to demonstrate that the scribe was somebody particular. He should also be able to read, write *and speak* Sumerian – a feat which only other scribes could appreciate.[9] He should know about bilingual texts, and know all the significations of the cuneiform signs (as syllabic signs and as ordinary word signs, and significations so secret that modern scholars do not know what the school text speaks about). And he should know about music and mathematics. All of it, believe it or not, was called *humanism* – nam-lú-ulù, Sumerian for “the state of being human”. Since lú corresponds to Latin *vir*, we might also translate *virtuosity*, on the condition of remembering the etymology of that word.

The ideological school texts do not specify the kind of mathematics that participated in scribal humanism. However, a look at the mathematics that was actually taught in school will allow us to make a reasoned guess.

Accounting documents and other economic texts contain numbers and inform us about metrologies, but beyond that they only confirm that the techniques created during the third millennium – not least Ur III – were still used. In order to find out about mathematical knowledge and thinking we have to go to the mathematical texts proper.

Three principal types can be distinguished: tables, ‘bare calculations', and problems. Beyond tables of reciprocals, products and fixed constants and metrological tables we find tabulations of squares and cubes and of numbers of the type $n^2\times(n+1)$. The ‘bare calculations' mostly concern the determination of products, squares and reciprocals (beyond those found in the standard tables). They are often made on the reverse of tablets whose obverse carries Sumerian proverbs or other pieces of literature; students of elementary calculation were thus already well advanced in the study of literature and language.

The advanced level *of mathematics* is found in the problem texts. These can be categorized in different ways. Some texts contain a single or a couple of problems, others a larger number. Of these latter, some are ‘theme texts', others ‘anthologies' containing a mixture of varied problems; it is noteworthy that no anthologies fuse mathematical problems with non-mathematical matters, not even numerology. *Mathematics* was thus conceived as a coherent and self-contained entity by the Old Babylonian scribes no less than by us.

Both anthologies and texts containing only a few problems usually indicate the procedure to be followed. Some theme texts do so too, others are ‘catalogues' that list only problem enunciations (perhaps together with the solution, which allowed control). We may assume

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[9] Really a feat. Sumerian and Babylonian are less similar than, say, Turkish and English.

that catalogues were meant to enable teachers to give different but similar problems to the students of a class.

Problems may be categorized according to the object dealt with or to the method applied. For instance, some problems regarding (say) the volume of a prismatic excavation ask only for the application of the formula for this volume; others require the application of 'algebraic' methods (to be explained below), still others involve methods that we are tempted to characterize as 'number theory'. It is therefore preferable to discuss first the basic techniques, those that serve real practice, and to look afterwards at the supra-utilitarian level (which take the basic techniques for granted).

The fundament for every practice was proportionality (not to be mixed up with proportion theory in Greek style!) and inverse proportionality. These, indeed, are presupposed by the tables of fixed constants: if one man can carry 540 bricks of a certain type 30 nindan in a day (Babylonian bricks were huge!), how many man-days are needed to carry N bricks of the same type M nindan? Geometric computations also made use of the fact that all linear extensions within a configuration of well-defined shape are proportional, and areas proportional to the square of one linear dimension; in both cases, the factors of proportionality could be found in the tables of fixed constants.

There was no concept of the angle as a measurable quantity but a distinction between practically right angles (those between sides that determine areas) and oblique (whence irrelevant) angles. Real surfaces were subdivided into practically right quadrangles and triangles; for quadrangles where measurement showed opposite sides not to be exactly equal, the 'surveyors' formula' (average length times average width) had been applied since the fourth millennium. For triangles, *the* length (that is, the relevant length, the one which was practically perpendicular to the width) was multiplied by the half of the width.[10]

Prismatic volumes were found as we would do it, with one proviso. While the basic unit for horizontal distance was the nindan, that for vertical distance was the kush ($={}^{1}/_{12}$ nindan, we remember). The basic unit for areas was thus the sar, the square nindan; but the same unit served for measuring volumes, surfaces being supposed to be provided with a 'virtual height' of one kush. In order to find a prismatic volume with base B and height h, B was therefore 'raised' (from 1) to (the real height) h; this being an operation of proportionality, the term 'raising' was generalized to serve for all multiplications based on proportionality – thus for all involving fixed constants.

At the boundary between the useful and the supra-utilitarian we find the 'Pythagorean theorem', obviously not known as a theorem – theorems do not exist in Babylonian mathematics – but quoted as a general rule in one text and used in a number of others.

The supra-utilitarian level proper had two main components (they do not exhaust the matter). One is the so-called 'algebra'.

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[10] It is a widespread fable that the Babylonians found the area of an arbitrary triangle as the product of two sides; that they did not is clearly demonstrated by field plans. But it was habitual to interpret a mathematical description as a reference to the simplest situation it might correspond to. The word we translate as "triangle" (actually rather "wedge") thus refers to a triangle whose area can be found fairly well as the product of its 'length' (if needed distinguished from 'the long length', the hypotenuse) and its semi-width; similarly, "length-width" was understood as a figure determined by a single length and a single width, i.e., a rectangle.

The ‘algebraic' problems speak of the sides and areas of squares and rectangles. When they were discovered around 1930 it was believed that the geometric idiom was accidental, that the sides were to be understood as mere numbers and the areas as products. However, accurate analysis of the vocabulary reveals that this reading is untenable: it cannot explain that the texts distinguish carefully between two different ‘additive' operations, two different ‘subtractions', two different ‘halves', and no less than four operations which the traditional reading conflates as ‘multiplication'; it also fails to understand certain recurrent phrases (which are therefore simply left out from translations), and it explains certain procedures badly. If instead we accept the geometric phrasing as essential, everything becomes clear.

Two problems about squares and two about rectangles make up the core of the discipline. Those dealing with rectangles form a group together with two that were already known in the Sargonic school ($\sqsubset\!\sqsupset(\ell,w)$ stands for the area of a rectangle with sides ℓ and w):

(1) $\sqsubset\!\sqsupset(\ell,w) = \alpha\ ,\ \ \ell = \beta$ (3) $\sqsubset\!\sqsupset(\ell,w) = \alpha\ ,\ \ \ell+w = \beta$

(2) $\sqsubset\!\sqsupset(\ell,w) = \alpha\ ,\ \ w = \beta$ (4) $\sqsubset\!\sqsupset(\ell,w) = \alpha\ ,\ \ \ell-w = \beta$.

After the Ur-III reform, (1) and (2) asked for nothing but application of metrological tables and tables of reciprocals and products. (3) and (4), on their part, are of the second degree, and so are the basic square problems

(5) $\square(s)+\alpha s = \beta$ (6) $\square(s)-\alpha s = \beta$

We may look at the simplest possible variant of (5), the first problem of the theme text BM 13901 (‘algebraic problems about one or more squares'). In literal translation of the words and with translated numbers it runs as follows:

1. I have heaped the surface and my confrontation: it is $^3/_4$. 1 the projection
2. you posit, half-part of 1 you break, make $^1/_2$ and $^1/_2$ hold,
3. $^1/_4$ to $^3/_4$ you join: alongside 1, 1 is equilateral. $^1/_2$ which you have made hold
4. from the body of 1 you tear out: $^1/_2$ is the confrontation.

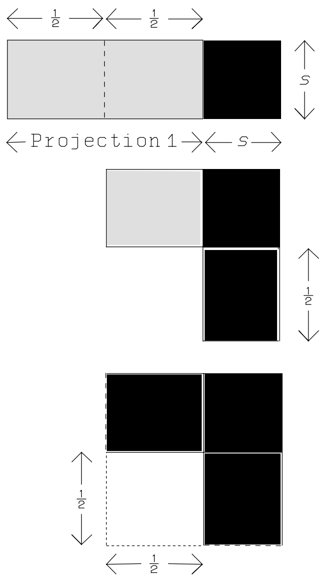

Figure 1. The transformations of the square area plus side.

Some terminological explanations are needed. A ‘confrontation', i.e., ‘a situation characterized by the confrontation of equals', denotes the quadratic configuration. Numerically it is parametrized by the side of the square. In agreement with the Euclidean definition of a ‘figure' (“that which is contained”), our square *is* its area (e.g., 9 m^2) and *has* a side (e.g., 3 m). The ‘confrontation', instead, *is* its side and *has* an area. ‘Heaping' is a symmetric additive operation which allows the addition of the numerical measures of entities of different kinds (e.g., lengths and areas, or days, workers and bricks). ‘Joining (to)', in contrast, is an asymmetric additive operation used only for operations that make sense concretely. The ‘projection' is a line of length 1; when applied perpendicularly to a line of length s it produces a rectangle with area $s\times1 = s$. The ‘half-part' is a ‘natural' half, a half that could be nothing but the half (the radius is the half-part of the diameter, the area of a right triangle is the product of the length and the half-part of the width); ‘breaking' is the operation which produces it. To make two lines a and b ‘hold' means to make them hold/contain a rectangle $\sqsubset\!\sqsupset(a,b)$; that ‘alongside S, r is equilateral' means that a surface S laid out as a square has the side r. ‘Tearing out' is a concrete subtractive operation, the reverse of ‘joining'; ‘tearing out'

p from T presupposes that p is a part of T – in the present case even 'of the body' of T.

We are thus told that the numerical sum of the area of a square (black in Figure 1) and its side is $^3/_4$. In order to provide this with a concrete interpretation, a 'projection 1' is applied to the left; this produces a (grey) rectangle whose area equals the side. Together with the black square, its area is thus $^3/_4$. Now, the projection (representing the rectangle) is broken into two, and the outer half-part moved around so as to contain together with the remaining half-part a square, whose area becomes $^1/_2 \times ^1/_2 = ^1/_4$. Joining this (white) square to the grey and black areas gives a completed square with area $^3/_4 + ^1/_4 = 1$ and thus with side $\sqrt{1} = 1$. Removal of the part that was moved around leaves $1 - ^1/_2 = ^1/_2$ for the (vertical) side of the original square.

The steps correspond well to our way to resolve the corresponding second-degree equation:

$$\begin{aligned} x^2 + 1\cdot x = \tfrac{3}{4} \quad &\Leftrightarrow \quad x^2 + 1\cdot x + (\tfrac{1}{2})^2 = \tfrac{3}{4} + (\tfrac{1}{2})^2 \\ &\Leftrightarrow \quad x^2 + 1\cdot x + (\tfrac{1}{2})^2 = \tfrac{3}{4} + \tfrac{1}{4} = 1 \\ &\Leftrightarrow \quad (x + \tfrac{1}{2})^2 = 1 \\ &\Leftrightarrow \quad x + \tfrac{1}{2} = \sqrt{1} = 1 \\ &\Leftrightarrow \quad x = 1 - ^1/_2 = ^1/_2 \end{aligned}$$

However, the analogy goes beyond the sequence of steps. As ours, the Babylonian procedure is *analytic*, that is, the unknown quantity is treated as if it was known, and through the steps thus made possible it is gradually extricated from the complex relation in which it was originally entangled. Further, the Babylonian method is *naive*, that is, it does not discuss why or under which conditions the transformations are valid. Obviously, we *can* transform the procedure into a 'critical' argument, an argument which discusses validity and conditions (Kant's very definition of *critique*), specifying for instance that the second-last step presupposes that we restrict ourselves to the domain of non-negative numbers; but usually we avoid this toil – in situations which we know well it is more economical to leave as obvious that which seems obvious. In ours as in the Babylonian case, the 'naive' character of the argument does not prevent it from being a full-blown argument; only the reading of the Babylonian texts as nothing but a sequence of numerical prescriptions (corresponding to reading only the right-hand sides of our equations) has given the impression that Babylonian mathematics was not reasoned but 'empirical' (whatever that is supposed to mean) or based on recipes found by 'trial and error'.

The procedure used for problems of type (5) also served for types (4) and (6); the trick of the 'projection' – used also in type (6) – creates a rectangle with known area and known difference between the sides, that is, type (4). Type (3) is solved by a different but analogous procedure (corresponding to Euclid's *Elements* II.5, whereas type (4) corresponds to II.6). Non-normalized problems were solved by means of proportional change of scale in one dimension – $\alpha\square(r)+\beta r = \gamma$ becoming $\square(\alpha r)+\beta(\alpha r) = \alpha\gamma$.

These methods, together with techniques for transforming linear relationships, constituted the 'elements' of Old Babylonian 'algebra'. They allowed *inter alia* to solve problems about 2, 3 or more squares with given linear relations between the sides and the areas, and even 'dual' problems where the sides were known and the coefficients of the relations unknown; they also solved biquadratic problems – in one text even a bi-biquadratic problem. Further, they served to solve non-geometric problems – a length might represent a price, a number of workers or a number in the table of reciprocals, and in biquadratic problems an area.[11] The geometry

[11] It should be taken note of that all such problems are exactly as useless in practice, that is, just as supra-utilitarian, as the algebraic area problems.

of measurable lines and areas thus constituted a neutral, functionally abstract representation. Apart from this representation being geometrical and not numerical, Old Babylonian ‘algebra' was thus *an algebra* in the same sense as the equation algebra of recent applied mathematics.

The starting point for this algebra appears to have been a few problems borrowed from the environment of lay surveyors, made at the moment when the scribe school adapted to the ideological conditions of the Old Babylonian period. The other main constituent of the supra-utilitarian level depended on the properties of the place value system; its starting point was thus the calculation tradition of the scribes proper, more precisely the shape it had taken on during Ur III.

A rather simple number play is the construction of products in sequence, for instance multiplying first 9 by 9, then again by 9, etc., until 9^{10} (but with no indication of the numerical rank of each product, that is, of the exponent). Simple (and barely supra-utilitarian) is also one of the techniques for finding reciprocals which have a finite reciprocal within the sexagesimal system (numbers of the type $2^p{\cdot}3^q{\cdot}5^r$, p, q and r being integer) but do not appear within the standard table: if $\tilde{n}$ is the reciprocal of n and $\bar{a}$ that of a (in practice, with $a = 2$, 3 or 5), then $\bar{a}\times\tilde{n}$ is the reciprocal of $a\times n$.

Much more sophisticated is another technique, which allows to find for example (a simple example!) the reciprocal of 1.51.6.40 (a floating-point number, we remember). From the standard table it is known that 6.40 is the reciprocal of 9; we therefore multiply by 9 in order to eliminate the two places; 16.40 results. This again ends in 6.40, and we therefore multiply with 9 once more. This yields 2.30. 1 51.6.40 is thus 2.30/(9×9), and since 2.30 is the reciprocal of 24, that of 1.51.6.40 is 24×9×9 = 32.24. Sophisticated but useless: the numbers chosen as fixed constants (that is, the divisors that might turn up in professional practice) were simple.

It was also possible to combine this ‘number theory' with ‘algebra'. Indeed, certain (artificial) problems about prismatic excavations could be reduced by algebraic methods to number problems; for instance, to find a number p such that either $p\times p\times(p+1) = 4{,}12$ or $p\times p\times(p+7) = 8$, or three numbers p, q and r such that $p\times q\times r = 0;36$, $p+q = 1$, $r = p+0;6$. The first problem is solved according to the text by means of a table “equilateral, one joined”; in the other cases, the solution must be guessed or facilitated by a factorization. Another combination of ‘algebra» and ‘number theory' is the famous tablet Plimpton 322. It uses pairs of mutually reciprocal numbers in order to create Pythagorean triplets, *perhaps* with the purpose of constructing resolvable algebraic second-degree problems.[12]

Neither at the utilitarian nor at the supra-utilitarian level is a single explicit theorem to be found. A few texts contain rules in abstract formulation;[13] all are close to the lay environment; it seems that the school gave up such rules as pedagogically inefficient. Instead, the prevailing teaching method was the solution of paradigmatic problems accompanied by

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[12] As pointed out to me by Eleanor Robson, it also appears from the formulation that the technique described above in purely numerical terms for finding difficult reciprocals was indeed based on the same area technique as the ‘algebra'.

[13] There is a tendency to confuse the two, but they are absolutely distinct. Take the ‘surveyors’ formula' for finding the area of an approximate rectangle: “multiply half of the sum of the lengths by half of the sum of the widths. This gives you the area”. This is a perfectly abstract rule, but no theorem in the sense of being an assertion provided with a proof; nor could it ever be, since it is false.

explanations; mostly, such explanations were oral, but a few written explicit instances[14] allow us to identify the traces of the oral instruction in certain others.

All the mathematical texts are school texts; even the tables were copied in school as a way to memorize them. The basic level certainly served the students (or some of them) later in their professional life. However, neither second-degree algebra nor the finding of difficult reciprocals could ever do that. Like Sumerian reading and writing these were matters that only other scribes (not least teachers) could appreciate; we must conclude that they are expressions of the *virtuosity* of scribes, of their particular *humanity*. This also explains why there was no incentive to develop a mathematics of demonstrated theorems. Only abilities that were formally akin to scribal professional duties could count as expressions of *scribal* virtuosity: *to write*, not only the Akkadian spoken by the employer but also the incomprehensible Sumerian language; and to make *calculations*, not only those needed for ascertaining measured justice – this idea was still an aspect of scribal identity – or requested by the boss but also such as went far beyond his understanding. To produce or prove theorems, on the other hand, had nothing to do with what a scribe was supposed to do and therefore could not express *scribal* virtuosity.

In our view, one does not become *a mathematician* by knowing (or teaching) how to apply mathematics, not even advanced mathematics. The mathematician is rather the one who *develops* mathematical knowledge. In this sense, the Old Babylonian teachers of mathematics were apparently not ‘mathematicians' but rather teachers of calculation, including ‘pure unapplicable' calculation. However, appearances might deceive, and for two reasons.

Firstly, there is the *construction* of problems. It is true that many problems can easily be fabricated backwards from a known solution. If we take a square of side $^1/_2$, it is obvious that the sum of the area and the side is $^3/_4$, and (once that trick is known) that the problem can be solved by quadratic completion. If we consider a rectangle with length 0;40 and width 0;30 it is also obvious that its area is 0;20, and (after some tedious calculation) that the rectangle contained by the diagonal and the cube on the length is 0;14,48,53,20. But it is *not* obvious that the sides of the original rectangle can be determined from these two magnitudes, that is, that the problem is bi-biquadratic. Only a systematic research passing through several intermediate steps – that is, only a *mathematician's* work – can produce that insight. But such mathematician's work was private work, and did not go into extant texts.

Secondly, precise analysis of the terminology reveals that different schools undertook, each in its own way, to create a canonical mode of exposition for mathematics and to establish the coherence of the mathematics they taught.[15] The purpose may well have been didactical (and perhaps to outrival competing schools); but even if the purpose was not the purity and coherence of mathematical knowledge, these efforts may still be characterized as mathematicians' efforts. After all, although its level was certainly higher, the goal of the Bourbaki group was quite similar: as the story goes, to take van der Waerden's presentation of Noether-Artin algebra in *Moderne Algebra* as a model first for the presentation of topology, then for that of the whole

[14] For instance, one which explains the trick of the ‘projection' and that of the quadratic completion.

[15] The introduction of the ‘projection' is one example; the lay surveyors had quietly joined lines to areas, presupposing that lines had a virtual width 1, just as surfaces had a virtual thickness. The schoolmasters, as Plato in *The Laws*, found the confusion unacceptable; three different schools invented each its own name for the entity.

of mathematics. Even though most Old Babylonian teachers of mathematics were not ‘mathematicians' (nor are all mathematics teachers today), mathematicians were not wholly absent from their world.

They only disappeared when the scribe school system collapsed together with the Old Babylonian kingdom and the idea that the state and the scribes it employed had to provide measured justice. But that is a different story.

La morale de cette histoire

What precedes is a synthetic portrait of a mathematical culture which, *as a culture*, is almost completely disconnected from ours – in spite of the fertile legacy of mathematical knowledge, techniques and notations which it has left to us. It draws on the original works of the author and other scholars who during the last 25 years have changed the picture of Mesopotamian mathematics thoroughly. Well enough – but what has that to do with mathematics education?

Admittedly, it has nothing to tell us about which mathematics should be taught, and does not help us much to teach better the mathematics which we actually teach. Certainly, the area technique of Old Babylonian ‘algebra' may be easier to grasp for some pupils than the symbolic manipulations of modern algebra; but it may be a dead end, not inviting to the general use of symbols which is the gist of post-Renaissance mathematics, and also of the much of the mathematics students will need later.

However, teaching mathematics also involves teaching *about* mathematics, and mathematics may be taught better if the teacher understands the subject-matter in broad context (not only more efficiently but also more responsibly). This may be a platitude, but even platitudes may be true. The present one (as many) only asks for being filled out concretely in order to become less trite. History may contribute to this concretization, provided we are speaking of true history, not of a mirror onto which contemporary mathematics projects its own formulae and in which it therefore sees nothing but itself. The portrait of the many levels of a whole mathematical culture should serve the purpose – originally, ‘concrete' meant ‘grown together', and that is exactly what the political, social, psychological, contentual and applicational aspects of a mathematical culture are. In the Mesopotamian case, we have seen how things which we regard as self-evident (abstract number, area linked to linear measure) are the outcome not only of generic ‘history' but of a process where they were bound up inseparably with the creation of a managerial state. We have also seen how mathematics, once created, could serve other purposes, and how different traditions could merge or inspire each other while being transformed in the process – etcetera.

It would be preposterous on the part of the historian to prescribe which aspects of the synthetical portrait should be made didactically fruitful. The portrait is one half of a bridge; the didactician and the teacher, knowing from proper experience, training and theory the terrain on their own side of the river, should construct the other half.

4. Bibliographic remarks

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(Nissen 1988) deals with the history of Mesopotamia in the fourth and third millennium, (Damerow & Englund 1987) with the Uruk metrologies, (Nissen, Damerow & Englund 1993) with the techniques of book-keeping until Ur III.

Concerning third millennium mathematics, one may consult (Friberg 1986; Høyrup 1982;

Foster & Robson 2004). The chief collections of Old Babylonian mathematical texts are (Neugebauer 1935–37; Neugebauer & Sachs 1945; Bruins & Rutten 1961). (Robson 1999) treats Ur-III and Old Babylonian practical mathematics and numerical calculation, (Høyrup 2002) Old Babylonian 'algebra', the lay tradition and its later influence. For the use of an abacus, see (Proust 2000).

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